A Noncommutative M–Theory Five–brane

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Abstract

We investigate, in a certain decoupling limit, the effect of having a constant C–field on the M–theory five–brane using an open membrane probe. We define an open membrane metric for the five–brane that remains non–degenerate in the limit. The canonical quantisation of the open membrane boundary leads to a noncommutative loop space which is a functional analogue of the noncommutative geometry that occurs for D–branes.

I. INTRODUCTION

The M–theory five–brane is a most mysterious object [1,2]. Several puzzles remain concerning its effective action and the theory of coincident five–branes, see e.g. [3]. Recently, there has been much interest in the noncommutative geometry that arises on D–brane worldvolumes from including a constant Neveu Schwarz two–form background potential [4–6]. It is

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natural to ask oneself whether similar deformed geometries play a role in M–theory. It has been suggested that the five–brane of M–theory plays the role of a D–brane in M–theory where, instead of an open string ending on the D–brane, we have an open membrane ending on the M5–brane [7,8]. We note that the open membrane ending on the five–brane can be reduced to the fundamental string ending on the D4–brane and by dimensional reduction one may therefore learn about the properties of D–branes by analysing the M5–brane, see for example [9]. The M–theory origin of the Neveu–Schwarz two–form background potential is given by the three–form potential C of eleven–dimensional supergravity. In order to investigate the occurrence of deformed geometries in M–theory one is therefore naturally led to consider an M–theory open membrane/five–brane system in the background of a constant C–field. The M5–brane in the background of a constant C–field has been investigated previously in the context of the AdS/CFT correspondence [10] and supersymmetric n–cycles [11].

It is known that, in order to derive the noncommutative properties of the D–brane, one may consider the interaction of open string end points with the Neveu–Schwarz two–form potential B_{NS} on the D–brane and perform a canonical quantisation [12,13]. Analogously, to determine the effect of including the background C–field on the usual five–brane geometry, we examine how an open membrane couples to an eleven–dimensional supergravity background with background five–branes. A preliminary analysis can be found in [12].

In this paper we will not perform the full canonical quantisation programme to the open membrane. Inspired by the case of D–branes [4,6], we will consider a certain decoupling limit in which (1) the bulk modes of the membrane decouple; (2) the dynamics of the boundary string in the five–brane is governed by the Wess–Zumino term of the M2–brane action; and (3) the worldvolume theory of the M–theory five–brane is described by a (2,0) supersymmetric field theory. Following this, we canonically quantise this Wess–Zumino boundary term only. The definition of the decoupling limit is more subtle than in the case of strings [6] because: (i) there is no coupling constant parameter g_s in M-theory; (ii) the M5–brane couples to the dual of the C–field; and (iii) the five–brane equations of motion

involve a non–linear algebraic self–duality condition which must be taken into account [2]. The canonical analysis of the boundary term leads to a noncommutative loop space on the five–brane worldvolume. This is a functional analogue of the noncommutative geometry that occurs for D–branes. Following this, we give a suggestion for a loop space star product. The decoupling limit mentioned above involves taking the field theory limit on the five–brane in the presence of a background field strength. We conjecture that the appropriate metric on the five–brane for describing this limit is not the usual induced closed membrane metric but rather an *open membrane metric*. This metric is analogous to the open string metric on a D–brane and indeed reduces to the latter upon double dimensional reduction. The open brane metrics are the natural metrics for describing the D–brane and the five–brane worldvolume theories in the presence of a background field strength in that it is these metrics rather than the induced ones that give the linearised mass shell condition on the worldvolume which in turn defines the field theory limits on the branes.

The structure of the paper is as follows. In Section II we introduce the open membrane/five–brane system. In Section III we discuss some properties of open brane metrics. Next, in Section IV, we give the decoupling limit and discuss its relation to the decoupling limit of [6]. In Section V we derive the deformed five–brane geometry. First we apply the Dirac canonical quantisation programme to the decoupled boundary Wess–Zumino term and derive the Dirac brackets of the five–brane coordinates. We then connect to the results in the literature for the D4–brane, examine the special case of the infinite momentum frame and finally give a star product for particular cases. Finally, in Section VI we give a discussion of our results and possible extensions to this work.

II. THE OPEN MEMBRANE/FIVE–BRANE SYSTEM

Our starting point is an open membrane in an eleven–dimensional supergravity background with background five–branes. The membrane boundary is a string that is constrained to lie within one of the five–branes which is separated from the stack of remaining five–branes.

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The reason that we include the additional stack of five–branes in the background is that it effectively stiffens the separated five–brane so that the membrane is not able to deform this five–brane and can be treated as a test membrane, as we shall discuss in more detail in Section IV.

The action for the open bosonic membrane is as follows (for the kappa-symmetric version of this action, see [14]):

$$S = S_k + \int_{M^3} f_2^* C + \int_{\partial M^3} f_1^* b \; , \tag{1}$$

where the kinetic term can be written in Polyakov form as

$$S_k = \frac{1}{2(\ell_p)^2} \int_{M^3} d^3\sigma \sqrt{-\det\gamma} \left(-\gamma^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N \hat{g}_{MN} + \ell_p^2\right) \; . \tag{2}$$

Here ℓ_p is the $D = 11$ Planck's constant, $\hat{g}_{MN}$ is the $D = 11$ spacetime metric and $\gamma_{\alpha\beta}$ is the auxiliary worldvolume metric. The maps f_2 and f_1 denote the embedding of the membrane and its boundary into the spacetime and the five–brane, respectively. The worldvolume three–form $f_2^* C$ is the pull–back of the $D = 11$ three–form potential C to the membrane worldvolume and, similarly, $f_1^* b$ is the pull–back of the five–brane two–form potential b to the boundary of the membrane. In terms of components, we write

$$(f_2^* C)_{\alpha\beta\gamma} = \partial_\alpha X^M \partial_\beta X^N \partial_\gamma X^P C_{MNP} \; , \quad (f_1^* b)_{ij} = \partial_i X^\mu \partial_j X^\nu b_{\mu\nu} \; , \tag{3}$$

where $M = 0, 1, \cdots, 9, 11$ are spacetime indices, $\mu = 0, 1 \cdots, 5$ are five–brane worldvolume indices, $\alpha = 0, 1, 2$ are membrane worldvolume indices and $i = 0, 1$ are indices on the boundary of the membrane.

Since the decoupling limit involves scaling Planck's constant ℓ_p we need to carefully assign dimensions to all quantities. We are using units in which the spacetime metric $\hat{g}_{MN}$, the worldvolume metric $\gamma_{\alpha\beta}$, all differential forms and the membrane worldvolume parameters σ^α are dimensionless. The spacetime coordinates X^M and the five–brane coordinates X^μ have dimension length. Note that the components C_{MNP} and $b_{\mu\nu}$ have dimension $(\text{mass})^3$ and $(\text{mass})^2$, respectively, in order for C and b to be dimensionless.

The coupling of b to the boundary of the membrane ensures that the open membrane action is invariant under the spacetime gauge transformations $\delta C = d\Lambda$ provided that $\delta b = -f_5^*\Lambda$, where f_5 denotes the embedding of the five–brane into spacetime. The two–form b satisfies the five–brane field equations. These are equivalent to a non–linear self–duality condition on the following gauge invariant three–form field strength of b:

$$\mathcal{H} = db + f_5^*C \ . \tag{4}$$

Here the last term is the pull–back of the spacetime three–form potential to the five–brane:

$$(f_5^*C)_{\mu\nu\rho} = \partial_\mu x^M \partial_\nu x^N \partial_\rho x^P C_{MNP} \ , \tag{5}$$

where $x^M(X^\mu)$ are local embedding functions satisfying the five–brane equations of motion. The non–linear self–duality condition on the five–brane reads [2]

$$\frac{\sqrt{-\det g}}{6}\epsilon_{\mu\nu\rho\sigma\lambda\tau}\mathcal{H}^{\sigma\lambda\tau} = \frac{1+K}{2}(G^{-1})_\mu{}^\lambda \mathcal{H}_{\nu\rho\lambda} \ , \tag{6}$$

where $\epsilon^{012345} = 1$ and the scalar K and the tensor $G_{\mu\nu}$ are given by

$$K = \sqrt{1 + \frac{\ell_p^6}{24}\mathcal{H}^2} \ , \tag{7}$$

$$G_{\mu\nu} = \frac{1+K}{2K}\left(g_{\mu\nu} + \frac{\ell_p^6}{4}\mathcal{H}_{\mu\nu}^2\right) \ . \tag{8}$$

We shall argue in the next section that the tensor $G_{\mu\nu}$ is the metric on the five–brane seen by an open membrane in the presence of a background three–form field strength $\mathcal{H}$. It is also understood that in the above three equations the indices are contracted with the induced five–brane metric:

$$g_{\mu\nu} = \partial_\mu x^M \partial_\nu x^N \hat{g}_{MN} \ . \tag{9}$$

We may parameterise the $D = 11$ spacetime by local coordinates $X^M = (X^\mu, Y^m)$, where $m = 6, 7, 8, 9, 11$ refer to the directions perpendicular to the five–brane, such that the Dirichlet condition on the membrane embedding fields becomes $Y^m(\sigma^\alpha)|_{\partial M^3} = 0$. The remaining

parallel embedding fields $X^\mu(\sigma^\alpha)$ obey a mixed Dirichlet and Neumann boundary condition. By varying the action (1), using $\delta Y^m|_{\partial M^3} = 0$ together with the definitions (4) and (9) one finds that [12]

$$\left[\frac{1}{\ell_p^2} \sqrt{-\det \gamma}\, n^\alpha \partial_\alpha X^M \hat{g}_{\mu M} + \epsilon^{\alpha\beta\gamma} n_\alpha \mathcal{H}_{\mu\nu\rho} \partial_\beta X^\nu \partial_\gamma X^\rho \right] \Bigg|_{\partial M^3} = 0 \ , \tag{10}$$

where n_α is the normal vector at the boundary. The first term is non–local from the point of view of the boundary string. As we shall see, this term drops out in the decoupling limit leaving a local field equation for the embedding fields $X^\mu(\sigma^i)$ of a closed string in six dimensions. The closed string boundary conditions are

$$X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma) \ , \tag{11}$$

where $\tau \equiv \sigma^0$ is the time coordinate and $\sigma \equiv \sigma^1$ is the spatial coordinate on the worldsheet. We shall consider backgrounds where $\mathcal{H}_{\mu\nu\rho}$ is constant. This is only consistent with (4) provided we require that the pull–back of the spacetime four–form field strength $F = dC$ to the five–brane vanishes, i.e. $f_5^\star F = 0$. After some manipulations (given in appendix A) we end up with the following action:

$$S = S_k + \int_{M^3} f_2^* \tilde{C} + \frac{1}{3} \int_{\partial M^3} d^2\sigma\, \mathcal{H}_{\mu\nu\rho} X^\mu \dot{X}^\nu X'^\rho \ , \tag{12}$$

where $\tilde{C}$ is the part of the three–form potential which is perpendicular to the five–brane, i.e. $f_5^\star \tilde{C} = 0$, and the dot and prime indicates differentiation with respect to τ and σ, respectively.

It will be useful to introduce a specific parametrization of the solutions of the self–duality condition (6) as follows[1] (see Appendix B):

$$\mathcal{H}_{\mu\nu\rho} = \frac{h}{\sqrt{1 + \ell_p^6 h^2}} \epsilon_{\alpha\beta\gamma} v_\mu^\alpha v_\nu^\beta v_\rho^\gamma + h\, \epsilon_{abc} u_\mu^a u_\nu^b u_\rho^c \ , \tag{13}$$

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[1]This parameterisation has been derived independently by [15].

$$G_{\mu\nu} = \frac{\left(1 + \sqrt{1 + h^2\ell_p^6}\right)^2}{4} \left(\frac{1}{1 + h^2\ell_p^6} \eta_{\alpha\beta} v_\mu^\alpha v_\nu^\beta + \delta_{ab} u_\mu^a u_\nu^b\right) . \tag{14}$$

Here h is a real field of dimension $(\text{mass})^3$ and (v_μ^α, u_μ^a), $\alpha = 0, 1, 2$, $a = 3, 4, 5$, are sechsbein fields in the nine–dimensional coset $SO(5,1)/SO(2,1) \times SO(3)$ satisfying

$$g^{\mu\nu} v_\mu^\alpha v_\nu^\beta = \eta^{\alpha\beta} , \quad g^{\mu\nu} u_\mu^a v_\nu^\beta = 0 , \quad g^{\mu\nu} u_\mu^a u_\nu^b = \delta^{ab} , \tag{15}$$

$$g_{\mu\nu} = \eta_{\alpha\beta} v_\mu^\alpha v_\nu^\beta + \delta_{ab} u_\mu^a u_\nu^b . \tag{16}$$

III. OPEN BRANE METRICS

The natural metric to examine the worldvolume of a D–brane with a non–trivial background Born–Infeld field strength $\mathcal{F}_{\mu\nu}$ is the so called open string metric. For a Dp–brane this metric is defined as follows:

$$G_{\mu\nu} = g_{\mu\nu} - \alpha'^2 \mathcal{F}_{\mu\rho} g^{\rho\lambda} \mathcal{F}_{\lambda\nu} , \qquad \mu = 0, 1, \cdots, p , \tag{17}$$

where $g_{\mu\nu}$ is the induced metric on the brane. The open string metric defines the mass shell condition for the open string modes propagating in the D–brane [6]. The expression (17) for the open string metric can also be read off from the low energy effective Dirac–Born–Infeld action which yields massless field equations with D'Alembertians $(G^{-1})^{\mu\nu}\nabla_\mu\nabla_\nu$ and Dirac operators $(G^{-1})^{\mu\nu}\Gamma_\mu\nabla_\nu$ (where ∇_μ contains the Christoffel symbol of $g_{\mu\nu}$). The decoupling limit on a Dp–brane as defined in [6] is non–degenerate in the sense that it yields a $(p+1)$–dimensional, massless field theory on the brane if the leading components of the open string metric form a rank $p + 1$ matrix.

In analogy with how branes appear as solutions in supergravity, the open string is visible as a supersymmetric worldvolume soliton solution to the massless D–brane field equations. It is instructive to examine the geometry of these so called BIon solutions [16] using the open string metric. A charged, static BIon, corresponding to the string ending on a Dp–brane,

excites one transverse scalar X and the time component of the Born-Infeld vector field A_μ. The solution is given by

$$(\alpha')^{\frac{1}{2}} A_0 = (\alpha')^{-\frac{1}{2}} X = H \ , \tag{18}$$

where H is a harmonic function on the transverse worldvolume space $\mathbb{E}^p$ given by:

$$H = 1 + \frac{Q}{r^{p-2}} \ , \qquad p > 2 \ . \tag{19}$$

After substituting this solution into the open string metric (17) we find the following two–block 0–brane line element:

$$ds^2(G) = -\frac{dt^2}{1 + \alpha'(\partial H)^2} + (dy^m)^2 \ , \qquad m = 1, \cdots p, \tag{20}$$

where $(\partial H)^2 = \delta^{mn} \partial_m H \partial_n H$ and $(dy^m)^2$ is the line element on $\mathbb{E}^p$. Taking the limit of large Q and carrying out the coordinate transformation $r^{2(p-2)} = \alpha' Q^2 / u^2$ we find the following metric:

$$ds^2(G) = \frac{r^2}{(p-2)^2} \left(\frac{-dt^2 + du^2}{u^2} + (p-2)^2 d\Omega_{p-1}^2 \right) \ . \tag{21}$$

This metric is conformally $AdS_2 \times S^{p-1}$. The ordinary induced metric $ds^2(g)$ is of three–block form with an additional line element proportional to dr^2 and therefore not conformally $AdS_2 \times S^{p-1}$ in the near horizon region.

The natural properties of the open string metric (17) in examining the open string spectrum, motivates the introduction of an analogous metric for studying an open membrane probing a five–brane in the presence of a background $\mathcal{H}$ field. We propose that the conformal class of this so called open membrane metric is given by the tensor $G_{\mu\nu}$ given in (8). This metric has been shown to dimensionally reduce to the open string metric on the D4–brane up to a scale factor [2]. Moreover, the massless field equations on the five–brane have kinetic terms involving the box operator $(G^{-1})^{\mu\nu} \nabla_\mu \nabla_\nu$ and the Dirac operator $(G^{-1})^{\mu\nu} \Gamma_\mu \nabla_\nu$ [2]. Further arguments in favour of the proposed open membrane metric are obtained by considering

other excitations than the massless tensor multiplet, like e.g. the self–dual string soliton corresponding to the open membrane ending on the five–brane [17][2]:

$$4\,\ell_p^2\,b_{01} = \ell_p^{-1}X = H\,, \qquad \mathcal{H}_{mnp} = \frac{\ell_p^{-2}}{4}\epsilon_{mnpq}\partial_q H\,, \qquad m = 2,3,4,5\,, \tag{22}$$

where H is a harmonic function on the transverse worldvolume space $\mathbb{E}^4$ given by:

$$H = 1 + \frac{Q}{r^2}\,. \tag{23}$$

Substituting this solution into (8) the line element of the open membrane metric has the structure of a two–block 1–brane solution:

$$ds^2(G) = \frac{1}{4}\left(1 + \sqrt{1 + \ell_p^2(\partial H)^2}\right)^2\left(\frac{-dt^2 + dx^2}{1 + \ell_p^2(\partial H)^2} + (dy^m)^2\right)\,, \tag{24}$$

whereas the induced metric has an additional line element proportional to dr^2. In the large Q limit the open membrane metric, upon making the coordinate transformation $r^4 = \ell_p^2 Q^2/u^2$, becomes:

$$ds^2(G) = \frac{u^2}{4}\left(\frac{-dt^2 + d\sigma^2 + du^2}{u^2} + 4d\Omega_3^2\right)\,, \tag{25}$$

which we identify as $AdS_3 \times S^3$ with a conformal factor.

IV. THE DECOUPLING LIMIT

We will now consider a limit of the open membrane/five–brane system with the following three properties. Firstly, the bulk modes of the membrane can be neglected. Secondly the boundary string that lives in the five–brane is governed solely by the Wess–Zumino term. Thirdly, the field theory limit on the five–brane is taken such that the physics on the five–brane is described by the self–dual tensor field theory. In order to demonstrate the limiting

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[2]Only in this section we use the index m to indicate the five–brane worldvolume directions transverse to the self–dual string. In all other sections the index m indicates the target space directions transverse to the five–brane.

procedure we require, it is instructive to first examine the the limit taken by Seiberg and Witten in deriving the noncommutative geometry of a D–brane from the open string.

The open string case

We study the decoupling limit at small string coupling g_s. The effective tensions τ of the string and the Dp–brane behave like $\tau_{\text{F1}} \sim 1$, $\tau_{\text{Dp}} \sim 1/g_s$. Therefore, for small g_s, the string is much lighter than the Dp–brane and can be treated as a test string probing the Dp–brane. Furthermore, the effective gravitational couplings $G_N \tau$ (Newton's constant times tension) behave like $G_N \tau_{\text{F1}} \sim g_s^2, G_N \tau_{\text{Dp}} \sim g_s$ and therefore we can assume that the spacetime background is approximately flat. The open string action reads

$$ S = \frac{1}{2\alpha'} \int_{M^2} d^2\sigma \, \bar{\partial} X^M \partial X^N \eta_{MN} + \frac{1}{2} \int_{\partial M^2} d\tau \mathcal{F}_{\mu\nu} X^\mu \dot{X}^\nu \,, \tag{26} $$

where $\mathcal{F}_{\mu\nu}$ is the background field strength on the Dp–brane. We assume that the only non–vanishing components of $\mathcal{F}$ are $\mathcal{F}_{r's'}$, where we have decomposed the worldvolume index μ as $\mu = (r, r')$ with $r = 0, 1, \cdots, p - \text{rank}\,\mathcal{F}$ and $r' = p + 1 - \text{rank}\,\mathcal{F}, \cdots, p$.

The string theory has closed string modes in the bulk, i.e. away from the D–brane. Their mass shell condition is governed by the closed string metric η_{MN}:

$$ \eta^{MN} p_M p_N = -\frac{4}{\alpha'}(N_{\text{L}} - 1) = -\frac{4}{\alpha'}(N_{\text{R}} - 1)\,. \tag{27} $$

Here $N_{\text{L,R}}$ indicates the oscillator number of the left and right movers and p_M is the bulk momentum of the closed string state. There are also open string modes propagating in the D–brane. An open string state with worldvolume momentum k_μ obeys the mass shell condition

$$ (G^{-1})^{\mu\nu} k_\mu k_\nu = -\frac{1}{\alpha'}(N^{\text{open}} - 1)\,, \tag{28} $$

where N^{open} is the open string oscillator number and the tensor $G_{\mu\nu}$ is the open string metric defined in (17). From the conditions on $\mathcal{F}_{\mu\nu}$, it follows that $G_{\mu\nu}$ is given by

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$$G_{\mu\nu} = \begin{cases} \eta_{\mu\nu} & \text{for} \quad \mu,\nu = r,s \\ \eta_{\mu\nu} - \alpha'^2 \mathcal{F}_{\mu\rho}\eta^{\rho\sigma}\mathcal{F}_{\sigma\nu} & \text{for} \quad \mu,\nu = r',s' \end{cases} \tag{29}$$

In order to describe the limit we first decompose the target spacetime index M as $M = (r, r', m)$ and split the spacetime metric η_{MN} into parts that are parallel and perpendicular to $\mathcal{F}$ and the D–brane as follows: $\eta_{MN} \rightarrow \eta_{rs} \oplus \eta_{r's'} \oplus \eta_{mn}$, where η_{rs} is perpendicular to $\mathcal{F}$ and parallel to the D–brane, $\eta_{r's'}$ is parallel to $\mathcal{F}$, and η_{mn} is perpendicular to the D–brane. The limit of Seiberg and Witten is obtained by taking $\epsilon \rightarrow 0$ such that[3]:

$$\eta_{r's'} \sim \epsilon \, \eta_{r's'} \,, \quad \alpha' \sim \epsilon^{\frac{1}{2}}\alpha' \,, \tag{30}$$

while keeping all other quantities fixed. The open string action scales as follows:

$$S \sim \frac{1}{2\epsilon^{1/2}\alpha'} \int_{M^2} d^2\sigma \, \bar{\partial}X^m \partial X^n \eta_{mn} + \frac{1}{2\epsilon^{1/2}\alpha'} \int_{M^2} d^2\sigma \, \bar{\partial}X^r \partial X^s \eta_{rs}$$

$$+ \frac{\epsilon^{1/2}}{2\alpha'} \int_{\Sigma} d^2\sigma \, \bar{\partial}X^{r'} \partial X^{s'} \eta_{r's'} + \frac{1}{2} \int_{\partial M^2} d\tau \mathcal{F}_{r's'} X^{r'} \dot{X}^{s'} \tag{31}$$

and the various string mass shell conditions become

$$\text{bulk} \quad : \quad \begin{cases} \eta^{mn}p_m p_n \,, \quad \eta^{rs}p_r p_s \sim -\frac{4}{\epsilon^{1/2}\alpha'}(N-1)\,, \quad N = N_{\mathrm{L}} = N_{\mathrm{R}}\,, \\ \eta^{r's'}p_{r'}p_{s'} \sim -\frac{4\epsilon^{1/2}}{\alpha'}(N-1)\,, \end{cases}$$

$$\text{D}-\text{brane} \quad : \quad (G^{-1})^{\mu\nu}k_\mu k_\nu \sim -\frac{1}{\epsilon^{1/2}\alpha'}(N^{\mathrm{open}}-1)\,,$$

where the open string metric is finite in this limit and is given by the maximal rank matrix

$$G_{\mu\nu} = \begin{cases} \eta_{\mu\nu} & \text{for} \quad \mu,\nu = r,s \\ -\alpha'^2 \mathcal{F}_{\mu\rho}\eta^{\rho\sigma}\mathcal{F}_{\sigma\nu} & \text{for} \quad \mu,\nu = r',s' \end{cases} \tag{32}$$

The limit (30) has the following three crucial properties that we wish to emulate for the open membrane/five–brane system:

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[3]In the equation below it is understood that the $\eta_{r's'}$ and α' occurring at the right–hand–side are ϵ–independent.

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i) The closed string bulk modes perpendicular to $\mathcal{F}$ with momentum $p_M = (p_r, 0, 0)$ or $p_M = (0, 0, p_m)$ are frozen out[4] thereby isolating the dynamics of the D–brane theory from the bulk.

ii) The closed string bulk modes with momentum $p_M = (0, p_{r'}, 0)$ give vanishing contribution to the action.

iii) On the D–brane all massive open string modes are frozen out and the decoupled, massless field theory on the brane is non–degenerate in the sense that it is $(p+1)$–dimensional and has finite effective coupling.

The noncommutative nature of the D–brane arises from quantising the remaining Wess–Zumino term.

The open membrane case

We now would prefer to proceed by analogy to the open membrane/five–brane system. However, the analysis of the decoupling limit for this system requires a slight modification, due to three circumstances: the absence of an analog of the string coupling constant; the fact that the open membrane probe couples to the three–form potential which is dual to the potential of the background five–brane; and the non–linear self–duality condition on the field strength in the five–brane worldvolume.

Thus, in M-theory, there is no sense in which the tension of an isolated five–brane can be said to be much larger than the tension of an isolated membrane. Neither can we assume an approximately flat spacetime background around a background five–brane. To prevent the membrane from deforming the background geometry, we therefore consider a $D = 11$ background consisting of a large stack of parallel five–branes, given by the solution

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[4] The massless, perpendicular bulk modes remain but their effective bulk coupling constant goes to zero.

$$ds^2(\hat{g}) = H^{-1/3}(dx^\mu)^2 + H^{2/3}(dy^m)^2 \ , \quad H = 1 + \frac{N_5\ell_p^3}{r^3} \ , \quad F = N_5\epsilon_4 \ , \tag{33}$$

where $\mu = 0, 1, \cdots, 5$; $m = 6, 7, 8, 9, 11$, N_5 is the number of stacked five–branes and ϵ_4 is the volume form on the transverse S^4. We let the open membrane end on one of these five–branes removed from the stack and placed at radius r_0. If $N_5 >> 1$ and r_0 is small, then the interactions between the stack and the separated five–brane effectively stiffens the latter so that the membrane can probe it without deforming it[5].

Under these conditions the induced metric on the five–brane (9) is given by:

$$g_{\mu\nu} = H^{-1/3}(r_0)\eta_{\mu\nu} \ . \tag{34}$$

Moreover, from (33) it follows that the $D = 11$ background four–form field strength satisfies $f_5^* F = 0$. From the discussion in Section II it follows that we may consider an open membrane action given by

$$S = \frac{1}{2(\ell_p)^2} \int_{M^3} d^3\sigma \sqrt{-\det\gamma} \left(-H^{-1/3}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - H^{2/3}\gamma^{\alpha\beta}\partial_\alpha Y^m \partial_\beta Y^n \delta_{mn} + \ell_p^2 \right)$$
$$+ N_5 \int_{M^3} f_2^\star \tilde{C} + \frac{1}{3} \int_{\partial M^3} d^2\sigma \, \mathcal{H}_{\mu\nu\rho} X^\mu \dot{X}^\nu X'^\rho \ , \tag{35}$$

where the $D = 11$ background three–form potential $\tilde{C}$ obeys $d\tilde{C} = \epsilon_4$ and $f_5^*\tilde{C} = 0$ and the background three–form field strength $\mathcal{H}_{\mu\nu\rho}$ on the five–brane is constant.

Unlike the case of the string ending on a D–brane, where the Neveu–Schwarz two–form potential can be tuned independently of the Ramond–Ramond potentials, the membrane couples to the dual potential $N_5\tilde{C}$ of the five–brane six–form potential, which becomes large in the limit of large N_5. However, $\tilde{C}$ affects only the perpendicular closed membrane bulk modes and does not prevent these modes from decoupling in the limit from the dynamics on the five–brane. We will therefore, from now on, drop the term in the open membrane action (35) containing $\tilde{C}$.

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[5] By analysing the Nambu–Goto action for the five–brane scalars, it can be shown that for large N_5 the characteristic length scale of five–brane deformations with large gradients is $N_5^{-\frac{1}{3}}\ell_p$.

Another difference from the string case is that we cannot with impunity take a scaling limit for $\mathcal{H}$, since the field equations for the five–brane impose the non–linear algebraic constraints (6). In order for our limit to be guaranteed to obey (6) we will use the explicit solution for $\mathcal{H}$ given by (13) and (16).

We propose the decoupling limit obtained by taking $\epsilon \to 0$ such that

$$\ell_p \sim \epsilon \, \ell_p \, , \tag{36}$$

$$\frac{N_5}{r_0^3} \sim \epsilon^{-3\delta} \frac{N_5}{r_0^3} \, , \tag{37}$$

$$h \sim \epsilon^{-\lambda} h \, . \tag{38}$$

For simplicity we shall assume that $\delta > 1$[6] such that we may drop the 1 from the harmonic function in the metric (33) [7]. It then follows from (34) that the induced five–brane metric and the sechsbein fields in (16) scales as

$$g_{\mu\nu} \sim \epsilon^{\delta-1} \, g_{\mu\nu} \, , \quad u_\mu^a \sim \epsilon^{\frac{1}{2}(\delta-1)} u_\mu^a \, , \quad v_\mu^\alpha \sim \epsilon^{\frac{1}{2}(\delta-1)} v_\mu^\alpha \, . \tag{39}$$

Furthermore, we assume that $\lambda \leq 3$. This implies that $h\ell_p^3$ remains finite which enables us to keep the three–form field strength and the open membrane metric non–degenerate in the limit. Thus we find that the open membrane action (35) scales as

$$S \sim \epsilon^{-\Delta} \left[\frac{1}{2\ell_p^2} \int_{M^3} d^3\sigma \sqrt{-\gamma} \left(-\epsilon^{\Delta+\delta-3} H^{-\frac{1}{3}} \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \epsilon^{\Delta-2\delta} H^{\frac{2}{3}} \gamma^{\alpha\beta} \partial_\alpha Y^m \partial_\beta Y^n \delta_{mn} + \epsilon^\Delta \ell_p^2 \right) \right.$$
$$\left. + \frac{1}{3} \int_{\partial M^3} d^2\sigma \, \mathcal{H}_{\mu\nu\rho} X^\mu \dot{X}^\nu X'^\rho \right] \, , \tag{40}$$

where we have defined

$$\Delta = \lambda - \frac{3}{2}(\delta - 1) \, . \tag{41}$$

We now make our requirements, as for the string case:

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[6] One can also consider the case $0 < \delta \leq 1$. This does not lead to any qualitative changes in the final result.

[7] We point out that the limit considered here differs from the Maldacena limit since energies are not kept fixed in the near horizon region.

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i) The perpendicular bulk modes Y^m are frozen out provided $\Delta - 2\delta < 0$.

ii) The action for the parallel bulk modes vanishes if $\Delta + \delta - 3 > 0$. (This amounts to the vanishing of the first term in (10) such that (10) turns into a local field equation on the string worldsheet.)

iii) We require a non–degenerate field theory limit on the five–brane. As discussed in Section III, we conjecture that the relevant metric is the open membrane metric (8). Thus we require the distances $ds^2(G)$ measured in the open membrane metric to scale more slowly than ℓ_p^2 and the leading part of the open membrane metric to be non–degenerate in the limit[8]:

$$G_{\mu\nu} \sim \epsilon^\beta \left(\bar{G}_{\mu\nu} + \mathcal{O}(\epsilon) \right) , \tag{42}$$

where

$$\beta < 2 , \qquad \det \bar{G}_{\mu\nu} \neq 0 . \tag{43}$$

From (14) it follows that the second condition in (43) requires $h\ell_p^3$ to remain finite in the limit which implies that $\lambda \leq 3$. The first condition in (43) amounts to $\delta < 3$.

Combining our assumption that $\delta > 1$ with the conditions obtained under (i)-(iii) we find the following restrictions on our parameters:

$$\Delta + \frac{3}{2}(\delta - 1) \leq 3 < \Delta + \delta , \quad \Delta < 2\delta , \quad 1 < \delta < 3 . \tag{44}$$

These conditions are solved by (Δ, δ) in a finite size region. For instance, $\delta = \frac{5}{3}$, $\lambda = 3$ and $\Delta = 2$ leads to a decoupled five–brane theory in a background with a non–linearly self–dual field strength, while $\delta = \frac{4}{3}$, $\lambda = 2$ and $\Delta = 2$ yields a linearly self–dual field strength.

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[8]This differs from the limit proposed in [6] where the rank of the open membrane metric is reduced from six to five in the limit.

A noteworthy feature is that (44) implies $\Delta > 0$, such that there is necessarily an overall scaling of the action in (40). Such a scaling was not required in the string case. A crucial difference between the string and the membrane is that only the string action (26) has a microscopic interpretation. On the other hand, the membrane action should be seen as an effective action. One interpretation of the scaling (40) with $\Delta > 0$ is that actually we are taking a semiclassical limit.

Summarizing, in order to understand the geometry of the five–brane worldvolume we are led to study the quantisation of the following action:

$$S = \frac{1}{3} \int_{\partial M^3} d^2\sigma \, \mathcal{H}_{\mu\nu\rho} X^\mu \dot{X}^\nu X'^\rho \, , \tag{45}$$

with $\mathcal{H}$ constant.

A comparison with the limits for a lifted D–brane

Before proceeding with the canonical analysis in the next section, we wish to compare the decoupling limit we propose for the M-theory five–brane with the one discussed in [6]. By applying the usual relations between string theory and M-theory

$$g_s = (R/\ell_p)^{3/2} \, , \quad \alpha' = \frac{\ell_p^3}{R} \, , \quad \mathcal{F}_{\mu\nu} = R \, \mathcal{H}_{\mu\nu 5} \, , \quad \mu, \nu = 0, 1, 2, 3, 4 \, , \tag{46}$$

one may lift the limits taken for the D4–brane to the case of the M5–brane. We assume that the rank of $\mathcal{F}$ is 4. This motivates the following limit for the M–theory five–brane on $\mathbf{R}^6$ [6]:

$$g_{IJ} \sim \epsilon \, g_{IJ} \, , \quad g_{55} \sim \epsilon^2 g_{55} \, , \quad g_{00} \sim -1 \, , \quad \ell_p \sim \epsilon^{1/2} \ell_p \, , \quad I = 1, 2, 3, 4 \, ,$$

$$\mathcal{H}_{012} \sim \epsilon^{-1} \sqrt{1+\epsilon} \, h_0 \, , \quad \mathcal{H}_{034} = -\epsilon^{-1} h \, , \quad \mathcal{H}_{125} \sim -h_0 \, , \quad \mathcal{H}_{345} \sim \sqrt{1+\epsilon} \, h \, , \tag{47}$$

with $h_0^2 - h^2 + \ell_p^6 h_0^2 h^2 = 0$ and $h_0 < 0$.

The presented constant three–form solution is a special case of (13) where a diagonalised $\mathcal{H}_{012} \oplus \mathcal{H}_{345}$ split background has been boosted in an ϵ dependent way in the $0-5$ direction

which turns on $\mathcal{H}_{034}$ and $\mathcal{H}_{125}$ components. This is required in order for the reduction along the fifth direction to give a rank 4 constant two–form solution on the D4–brane. Equivalently, we can reduce along a skew spacelike direction in the $0 - 5$ plane instead of reducing along the fifth direction to obtain a rank 4 solution on the D4–brane.

By construction, the dimensional reduction of this solution and limit will give the appropriate limit for the D4–brane. However, under this limit the open membrane metric (14) behaves like $G_{55}\ell_p^{-2} \to 0$. This means that our condition (iii) for the field theory limit (involving the open membrane metric) on the five–brane to be valid is not satisfied. Obviously, this is not a problem for the compactified theory where the excitations in the fifth direction are suppressed.

V. CANONICAL ANALYSIS

In this section we will canonically quantise the action (45) with constant field strength $\mathcal{H}_{\mu\nu\rho}$. In the first part of this section we will assume that the field strength can be diagonalised as follows (see Appendix B for details) [6]:

$$\mathcal{H}_{012} = -\frac{h}{\sqrt{1 + \ell_p^6 h^2}} \, , \qquad \mathcal{H}_{345} = h \, . \tag{48}$$

where the dimensionless tensor multiplet "coupling" $h\ell_p^3$ is non–vanishing provided the decoupling limit (38) has been taken with $\lambda = 3$. For $\lambda < 3$ the limit results in a linear tensor multiplet. At the end of this section we discuss the case in which the field strength cannot be brought into the above form (see Appendix C for details).

In the parameterisation (48) the action (45) splits into two independent Lagrangians for the two sets of coordinates $X^{0,1,2}$ and $X^{3,4,5}$:

$$S = \frac{h}{3\sqrt{1 + \ell_p^6 h^2}} \int_{\partial M^3} d^2\sigma \, \epsilon_{\alpha\beta\gamma} X^\alpha \dot{X}^\beta X'^\gamma + \frac{h}{3} \int_{\partial M^3} d^2\sigma \, \epsilon_{abc} X^a \dot{X}^b X'^c \, , \tag{49}$$

where $\alpha = 0, 1, 2$ and $a = 3, 4, 5$. The action is invariant under worldsheet reparameterisations:

$$\delta_\xi X^\alpha = \xi^i \partial_i X^\alpha \ , \quad \delta_\eta X^a = \eta^i \partial_i X^a \ , \qquad i = 0, 1 \ . \tag{50}$$

Note that, due to the absence of a worldsheet metric, there is no need to identify the vector fields ξ and η. The equations of motion are:

$$\epsilon_{\alpha\beta\gamma} \dot{X}^\beta X'^\gamma = 0 \ , \quad \epsilon_{abc} \dot{X}^b X'^c = 0 \ . \tag{51}$$

This means that the embedding of the worldsheet in the 0,1,2 directions is a one–dimensional submanifold of that space. And similarly the embedding in the 3,4,5 directions is also one–dimensional.

A special feature of our system is that both the equations of motion and the gauge transformations are first order in derivatives. Thus, before we start the canonical analysis, it is instructive to first analyse the solutions to the equations of motion. We assume that the one–dimensional embedding of the worldsheet in the 0,1,2 directions is timelike. This means that we can fix the following static gauge for the ξ symmetry:

$$X^0 = \tau \ . \tag{52}$$

The field equation then implies that

$$X'^{1,2} = 0 \ . \tag{53}$$

There are two inequivalent sectors of the theory which differ by how the one–dimensional embedding in the remaining 3,4,5 directions takes place.

i) The string sector, for which the gauge fixed X^a field equation is

$$\dot{X}^a = 0 \ . \tag{54}$$

ii) The particle sector, for which the gauge fixed X^a field equation is

$$X'^a = 0 \ . \tag{55}$$

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Hence, in the case of (i) the embedded worldsheet is a two–dimensional surface which is unconstrained in the $X^{1,2}$ directions and fixed in the $X^{3,4,5}$ directions. In the case of (ii) the worldsheet is contracted into a one–dimensional worldline which is unconstrained in all five spatial directions. (By unconstrained we mean that the coordinate is pure gauge; this will become apparent after the analysis described below.) In the case of strings ending on D–branes the analogous field equations are much simpler, namely: $\dot{X}^i = 0$, $i = 1, \cdots, p$. This leads to the notion of a zero–brane in the D–brane. Here however we find a membrane in the five–brane due to the additional Dirichlet conditions (54). The boundary string is positioned in the space transverse to this membrane.

Let us continue by analysing the phase space dynamics of the three Euclidean coordinates $\vec{X} = (X^3, X^4, X^5)$. The canonical momenta are given by

$$\Pi_a(\sigma) := \frac{\delta S}{\delta \dot{X}^a(\sigma)} = -\frac{h}{3}\epsilon_{abc}X^b X'^c . \tag{56}$$

The non–trivial canonical Poisson brackets are:

$$\{X^a(\sigma), \Pi_b(\sigma')\} := \delta^a_b \, \delta(\sigma - \sigma') . \tag{57}$$

Since the Lagrangian is first order in time derivatives there is one primary constraint $\phi_a(\sigma)$ for each canonical momentum, given by

$$\phi_a := \Pi_a + \frac{h}{3}\epsilon_{abc}X^b X'^c \approx 0 . \tag{58}$$

The canonical Hamiltonian $H := \vec{\Pi} \cdot \dot{\vec{X}} - L$ vanishes. Instead, one introduces a generalised Hamiltonian $H_{\text{gen}} := \int d\sigma \, \lambda^a(\sigma)\phi_a(\sigma)$ where $\lambda^a(\sigma)$ are three Lagrange multipliers. To proceed with the canonical analysis we study the consistency conditions

$$\dot{\phi}_a(\sigma) := \{H_{\text{gen}}, \phi_a(\sigma)\} = \lambda^b(\sigma)M_{ba}(\sigma) \approx 0 , \tag{59}$$

where

$$\{\phi_a(\sigma), \phi_b(\sigma')\} = M_{ab}(\sigma)\delta(\sigma - \sigma') , \quad M_{ab} = h \, \epsilon_{abc}X'^c . \tag{60}$$

Clearly, (59) imposes no further phase space constraints. It simply sets to zero the Lagrange multipliers in the directions where the matrix M_{ab} is non–degenerate. These directions correspond to second class constraints. In a direction where M_{ab} is degenerate the Lagrange multiplier remains undetermined and such a direction corresponds to a first class constraint. Hence, if $|\vec{X}'| \equiv 0$ then there are three first class constraints and the phase space is trivial. On the other hand, provided that $|\vec{X}'| \neq 0$ the matrix M_{ab} has only one zero eigenvector, given by $\lambda^a(\sigma) = \lambda(\sigma)X'^a$. The matrix M_{ab} is therefore non–degenerate in the two–dimensional subspace orthogonal to $\vec{X}'$. Thus one introduces a projection onto this subspace as follows $(I = 1, 2)$:

$$P_I{}^a(\sigma)P_J{}^b(\sigma)\delta_{ab} = \delta_{IJ} \ , \tag{61}$$

$$\delta^{IJ}P_I{}^a(\sigma)P_J{}^b(\sigma) = \delta^{ab} - \frac{X'^a X'^b}{|\vec{X}'|^2} \ , \tag{62}$$

$$\epsilon^{IJ}P_I{}^a(\sigma)P_J{}^b(\sigma) = \frac{\epsilon^{abc}X'^c}{|\vec{X}'|} \ . \tag{63}$$

The three constraints ϕ_a now split into the two second class constraints

$$\chi_I := P_I{}^a\phi_a \ , \tag{64}$$

with the now non–degenerate matrix

$$\{\chi_I(\sigma), \chi_J(\sigma')\} := M_{IJ}(\sigma)\delta(\sigma - \sigma') \ , \qquad M_{IJ} = P_I{}^a P_J{}^b M_{ab} \ , \tag{65}$$

and one first class constraint

$$\phi := X'^a\phi_a \equiv X'^a\Pi_a \ , \tag{66}$$

which acts as the generator of σ–reparameterisations. The resulting generalised Hamiltonian,

$$H_{\text{gen}} = \int d\sigma \, \lambda(\sigma)\phi(\sigma) \ , \tag{67}$$

leads to the canonical field equations

$$\dot{X}^a(\sigma) = \lambda(\sigma)X'^a(\sigma) \ , \tag{68}$$

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which are indeed consistent with the Lagrangian field equations (51) when $|\vec{X}'| \neq 0$. We remark that the first class nature of the constraint ϕ means that the gauge invariant dynamics is trivially realised (i.e. $\dot{V} = 0$) on reparameterisation invariant functionals $V[\vec{X}]$ satisfying

$$\int d\sigma\, \eta(\sigma) X'^a(\sigma) \frac{\delta V}{\delta X^a(\sigma)} \approx 0 \ . \tag{69}$$

The analysis of the X^α coordinates is similar, but we require this system to admit the solution $X^0 = \tau$. This implies, from the equations of motion, that we must consider the trivial sector $X'^\alpha \equiv 0$. (In this sector $X^0 = \tau$ is a gauge choice for the constraint $\phi_0 \approx 0$.) To summarise, the two admissible sectors of the theory given above are: (i) $X'^\alpha = 0$ and $|\vec{X}'| = 0$; and (ii) $X'^\alpha = 0$ and $|\vec{X}'| \neq 0$.

Let us continue by deriving the symplectic structure of the phase space. In the 0,1,2 directions there are no second class constraints and we find vanishing brackets between the $X^{0,1,2}$ coordinates. We next turn to the $X^{3,4,5}$ coordinates. In the string sector, the presence of the second class constraints leads us to define a Dirac bracket as follows:

$$[A, B]^D := \{A, B\} - \int d\sigma\, \{A, \chi_I(\sigma)\} (M^{-1})^{IJ}(\sigma) \{\chi_J(\sigma), B\} \ . \tag{70}$$

Here A and B are general phase space variables and $(M^{-1})^{IJ}$ is the ordinary matrix inverse of the matrix M_{IJ} defined in (65). The basic Dirac brackets between the coordinates X^a and the first class constraint ϕ are:

$$[X^a(\sigma), X^b(\sigma')]^D = -\frac{1}{h} \frac{\epsilon^{abc} X'^c(\sigma)}{|\vec{X}'(\sigma)|^2} \delta(\sigma - \sigma') \ , \tag{71}$$

$$[X^a(\sigma), \phi(\sigma')]^D = X'^a(\sigma)\delta(\sigma - \sigma') \tag{72}$$

$$[\phi(\sigma), \phi(\sigma')]^D = 2\phi(\sigma)\delta'(\sigma - \sigma') + \phi'(\sigma)\delta(\sigma - \sigma') \ . \tag{73}$$

From the point of view of the five–brane the fields X^a are worldvolume coordinates. The Dirac bracket (71) therefore describes a noncommutative loop space on the five–brane. Finally, we discuss three special topics: dimensional reduction, the canonical analysis in the infinite momentum frame and the Moyal quantisation in case of a compact direction.

Double Dimensional Reduction

We wrap the five–brane and the membrane around a spacetime circle of radius R. In the limit of small R the five–brane becomes a D4–brane and the wrapped membrane becomes a fundamental string. Thus one recovers the case of a string ending on a D4–brane in the background of constant Neveu–Schwarz two–form potential. Explicitly, taking winding numbers into account, we find for a five–brane winding M times in spacetime and a string winding N times in the five–brane

$$X^5(\sigma) = NR\sigma\,, \qquad \mathcal{H}_{\mu\nu 5} = \frac{M}{R}\mathcal{F}_{\mu\nu}\,, \tag{74}$$

where the index μ now refers to the D4–brane worldvolume and $\mathcal{F}$ is normalised such that it obeys $d\mathcal{F} = H^{NS}$ where H^{NS} is the $D = 10$ Neveu–Schwarz three–form field strength. The membrane winds MN times in spacetime and the membrane action (1) reduces to MNS_{F1}, where S_{F1} is the $D = 10$ string action (26). For illustrative purposes, let us choose $\mathcal{H}_{345} = h$ and $\mathcal{H}_{012} = -h/\sqrt{1 + \ell_p^6 h^2}$. For this choice we find a rank two magnetic field[9]

$$\mathcal{F}_{34} = \frac{hR}{M}\,, \tag{75}$$

leading to a noncommutative D–brane worldvolume with

$$[X^3, X^4] = \frac{1}{MN}(\mathcal{F}^{-1})^{34}\,. \tag{76}$$

Alternatively, the double dimensional reduction can be performed directly at the level of the algebra (71)-(73) by fixing the gauge $X^5(\sigma) = NR\sigma$. The gauge fixed Dirac bracket $[\cdot,\cdot]^{D'}$ reads

$$[X^3(\sigma), X^4(\sigma')]^{D'} = -\frac{1}{hNR}\delta(\sigma - \sigma')\,. \tag{77}$$

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[9]This result can be generalized to a rank 4 magnetic field by either reducing over a skew spacelike direction in the $0 - 5$ plane or boosting the constant $\mathcal{H}$ background in the $0 - 5$ direction turning on $\mathcal{H}_{034}$ and $\mathcal{H}_{125}$ components, and then reduce over the fifth direction.

This bracket reduces to (76) provided we make use of (75) and identify $X^{3,4}$ with the zero–modes $X^{3,4} = \int d\sigma X^{3,4}(\sigma)$ of the doubly reduced string.

Canonical Analysis in the infinite momentum frame

In the infinite momentum frame discussed in Appendix C the field strength $\mathcal{H}$ is degenerate along a null direction:

$$\mathcal{H}_{-\mu\nu} = 0 \ . \tag{78}$$

The remaining components of the field strength are given by (see Eq. (C8) in Appendix C):

$$\mathcal{H}_{+pq} = F^-_{pq} = -\frac{1}{2}\epsilon_{pqrs}F^-_{rs} \ , \quad \mathcal{H}_{pqr} = 0 \ , \quad p = 1,2,3,4 \ . \tag{79}$$

Thus the lightcone coordinate X^- is decoupled from the remaining fields X^+ and X^p in the string action (45). The action has the reparameterisation invariance:

$$\delta_\eta X^+ = \eta^i \partial_i X^+ \ , \quad \delta_\eta X^p = \eta^i \partial_i X^p \ . \tag{80}$$

In the non–trivial sector the embedding of the worldsheet is a two–dimensional surface and the gauge fixed solutions to the field equations can be taken to be

$$X^- = \sigma^- \ , \quad \partial_- X^+ = \partial_- X^p = 0 \ , \tag{81}$$

where $\sigma^\pm = \frac{1}{\sqrt{2}}(\tau \pm \sigma)$. To give a phase space formulation, we introduce canonical momenta with respect to two–dimensional lightcone velocities as follows:

$$\Pi_\mu(\sigma^+) := \frac{\delta S}{\delta \partial_- X^\mu(\sigma^+)} \ , \quad \mu = +, p \ . \tag{82}$$

The first order nature of the Lagrangian implies that there are five primary constraints

$$\phi_\mu := \Pi_\mu + \frac{1}{3}\mathcal{H}_{\mu\nu\rho}X^\nu \partial_+ X^\rho \approx 0 \ , \quad \mu = +, p \ . \tag{83}$$

Since F^-_{pq} has rank four, it follows that the bracket $\{\phi_\mu, \phi_\nu\}$ has rank four and zero direction given by $\partial_+ X^\mu$. Hence there is one first class constraint, that we can choose to be

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$$\phi := \partial_+ X^\mu \phi_\mu \equiv \partial_+ X^\mu \Pi_\mu \ , \quad \mu = +, p \ , \tag{84}$$

and four second class constraints χ_I, $I = 1, 2, 3, 4$. Provided that $\partial_+ X^+ \neq 0$, we can choose these to be

$$\chi_I = \delta_I^p \phi_p \ . \tag{85}$$

The basic Dirac brackets are found to be

$$[X^p(\sigma_1^+), X^q(\sigma_2^+)]^D = \frac{[(F^-)^{-1}]^{pq}}{\partial_+ X^+} \delta(\sigma_1^+ - \sigma_2^+) \ ,$$

$$[X^+(\sigma_1^+), X^\mu(\sigma_2^+)]^D = 0 \ , \quad \mu = +, p \ , \tag{86}$$

and the analogs of (72) and (73).

In a background with a compact light–like direction of radius R, the gauge choice $X^+ = NR\sigma^+$ leads to a local, field independent bracket analogous to (77):

$$[X^p(\sigma_1^+), X^q(\sigma_2^+)]^D = \frac{[(F^-)^{-1}]^{pq}}{NR} \delta(\sigma_1^+ - \sigma_2^+) \ . \tag{87}$$

A double dimensional reduction along the compact direction gives the Dirac bracket for the rank 4 noncommutative D4–brane with Born-Infeld field strength $\mathcal{F}_{\mu\nu} = \frac{R}{M} \delta_\mu^p \delta_\nu^q F_{pq}^-$ where M is the number of times the five–brane winds around the compact direction.

Moyal quantisation

Usually after deriving the noncommutative structure one determines the star product. This cannot be done in a direct way for the bracket (71). We will discuss a proposal for how this might be done in the following discussion section. However, in the special case where the five–brane has a compact spacelike direction, the gauge fixed Dirac brackets (77) are straightforward to quantise by introducing the following Moyal product ($I, J = 3, 4$):

$$F(X) \star G(X) = \exp\left[-\frac{1}{2hNR} \int d\sigma \epsilon^{IJ} \frac{\delta}{\delta X^I(\sigma)} \frac{\delta}{\delta Y^J(\sigma)} \right] F(X) G(Y)|_{X=Y} \ , \tag{88}$$

where F and G are functionals of $X^{3,4}$.

Similarly, in a background with a compact light–like direction of radius R, the Moyal quantised star–product, based upon the brackets (87), is given by $(p = 1, 2, 3, 4)$:

$$F[X] \star G[X] = \exp\left[-\frac{1}{2NR}\int d\sigma^+ \, [(F^-)^{-1}]^{pq}\frac{\delta}{\delta X^p(\sigma^+)}\frac{\delta}{\delta Y^q(\sigma^+)}\right]F[X]G[Y]|_{X=Y} \, , \quad (89)$$

where F and G are functionals of X^p.

VI. DISCUSSIONS

In this paper we have studied the effect of having a constant C–field on the M–theory five–brane using an open membrane probe. We proposed a specific decoupling limit in which the open membrane metric remains non–degenerate. An unconventional feature of this decoupling limit, not encountered in the case of a string probing a D–brane, is that the open membrane action scales with an overall factor, as was discussed in Section IV.

We hope that our work provides the motivation and initial steps to study noncommutative loop algebras. A natural question to ask in this context is whether we can define a star product like in the case of D–branes. The utility of the star product is that to incorporate the effects of the noncommutativity of the space on fields one simply replaces ordinary products with star products to give the deformed theory.

In the case of a point particle moving in a finite dimensional Poisson manifold M with Poisson structure Ω^{ij}, one may in principle always find local canonical coordinates in which Ω^{ij} is a constant matrix and the star product then becomes simple. The reparameterisation independent definition of the star product was first given by Kontsevich [18] and has recently been cast into a more physical context by the work of [19]. The latter work makes use of a path integral representation of the star product.

There is a natural extension of the definition of star product given in [19] that applies to the case of the open membrane/five–brane system. A physically intuitive definition of the star product of two loop functionals $F[X]$ and $G[X]$ is given by the following path integral expression

$$(F \star G)[X(\sigma)] = \int\limits_{\substack{\lim\limits_{\tau \to \pm\infty} X(\tau,\sigma)=X(\sigma)}} \mathcal{D}X \; F[X(\tau=1,\sigma)] \; G[X(\tau=0,\sigma)] \; \exp\frac{i}{\hbar}S[X] \;, \quad (90)$$

with the action S given by (45). In case the five–brane winds around a compact direction in the background we expect (90) to reproduce the Moyal products given in (88) and (89). It would be interesting to see whether the proposed path integral representation of the deformed five–brane geometry will facilitate the construction of a loop space analog of the star product of ordinary fields.

It will be desirable to represent the loop algebra also on ordinary fields such as those of the massless $(2,0)$ tensor multiplet on the five–brane. One possible direction for such constructions may be to introduce so called loop space covariant derivatives [20], which naturally incorporates the two–form potential in the tensor multiplet.

The main motivation for introducing a noncommutative loop algebra in this work was provided by a study of the M–theory five–brane. However, there are also other reasons to study the same system. For instance, the action (45) is formally the same as the action for a string interacting with a linear background Neveu–Schwarz two–form. Here we identify $B^{\text{NS}}_{\mu\nu} = \frac{1}{3}\mathcal{H}_{\mu\nu\rho}X^\rho$ such that the Neveu–Schwarz field strength, $H^{\text{NS}} = dB^{\text{NS}}$ is a constant. However, despite the resemblance of the terms in the action, the conditions on the background fields differ. In the previous case $\mathcal{H}$ must obey the field equations of the five–brane, in the case of the string with a background Neveu–Schwarz field strength, the background fields must obey the Einstein's equations for the appropriate supergravity. One can construct a solution with constant Neveu–Schwarz field strength, but the space is necessarily curved and one must again take care when discussing limits.

It is interesting that the action (45) also arises from the dimensional reduction of the higher order Chern–Simons term that appears in 11–dimensional supergravity $S = \int_{M^{11}} C \wedge H \wedge H$. More precisely, take $M^{11} = M^2 \times M^9$ and then reduce on M^9 as follows: Let $C = X^I \gamma_I$ where $\gamma_I \in H^3(M^9)$, $I = 1 \cdots b^3(M^9)$ and X^I are scalars on M^2, one then identifies H as the triple intersection form on M^9, $H_{IJK} = \int_{M^9} \gamma_I \wedge \gamma_J \wedge \gamma_K$. For this case the only constraints on the background field H is that it is a triple intersection form. The kinetic terms of the

supergravity will introduce period matrices of $\{\gamma_I\}$ that will play the role of a metric for X^I. In 5–dimensional supergravity there is an analogous treatment of the $A_1 \wedge F_2 \wedge F_2$ term. It would be interesting to see whether (compactified) supergravity theories allow limits in which only the Wess–Zumino terms survive.

It is worth remarking that in the decoupling limit we defined in this work, it appears possible to take seriously a quantisation program for the membrane. This is because one has decoupled the bulk modes and so one is left with only the boundary string in the five–brane. It would be interesting to see how far one can carry out the quantisation of such a system. In doing this one should use the full supersymmetric branes.

Finally, it is known that for the case of the D–brane the noncommutative U(1) theory is non–abelian. It is our hope that the development of the noncommutative M5–brane might shed light on the as yet unknown non–abelian structure of the five–brane worldvolume theory.

Note Added: After this paper appeared, we received the work of [21] in which related ideas are discussed from a different point of view.

ACKNOWLEDGEMENTS

APPENDIX A: THE WESS–ZUMINO TERM OF THE MEMBRANE

In this appendix we derive the form of the Wess–Zumino term of the membrane as given in (12).

Assume that $f_5^* F = 0$ and $\mathcal{H}$ is constant. Then we can write $f_5^* C = dc$ and $\mathcal{H} = d(b+c)$, at least locally. The two Wess–Zumino terms in (1) can therefore be written as a Wess–Zumino term for the bulk modes perpendicular to the five–brane plus a string Wess–Zumino term as follows:

$$\int_{M^3} f_2^* \tilde{C} + \int_{\partial M^3} f_1^*(b+c) \; , \tag{A1}$$

where $\tilde{C}$ obeys $f_5^* \tilde{C} = 0$. To see this, one writes $C = \tilde{C} + dC_2$ where $f_5^* C_2 = c$. One then applies Stokes' theorem to $\int_{M^3} C$ and uses the identity

$$f_2^\star C_2 |_{\partial M^3} = f_1^\star f_5^\star C_2 = f_1^\star c \, . \tag{A2}$$

Finally, one uses the fact that, since $\mathcal{H}$ is constant, we have the identity

$$(b+c)_{\mu\nu} = \frac{1}{3} \mathcal{H}_{\mu\nu\rho} X^\rho \, . \tag{A3}$$

Combining everything then leads to the form (12) of the Wess–Zumino term for the string in the five–brane.

APPENDIX B: THE $SO(5,1)/SO(2,1) \times SO(3)$ PARAMETERISATION OF $\mathcal{H}_{\mu\nu\rho}$

In this appendix we parameterise a generic solution to the non–linear self–duality condition (6) using a real scale factor and a coset element in $SO(5,1)/SO(2,1) \times SO(3)$. The special solutions corresponding to an infinite momentum frame are described in the next appendix. The non–linear self–duality condition (6) is equivalent to a linear self–duality condition [2]

$$\mathcal{H}_{\mu\nu\rho} = 4(m^{-1})_\mu{}^\lambda h_{\nu\rho\lambda} \, , \quad h_{\mu\nu\rho} = \frac{\sqrt{-g}}{6} \epsilon_{\mu\nu\rho\sigma\lambda\tau} h^{\sigma\lambda\tau} \, ,$$

$$m_{\mu\nu} = g_{\mu\nu} - 2\ell_p^6 k_{\mu\nu} \, , \qquad k_{\mu\nu} = h_{\mu\lambda\rho} h_\nu{}^{\lambda\rho} \, . \tag{B1}$$

It can be shown that the matrix $k_{\mu\nu}$ is traceless and that its square is proportional to $\eta_{\mu\nu}$. Hence $k_{\mu\nu}$ can be written as

$$k_{\mu\nu} = \lambda_+ v_\mu^+ v_\nu^+ + \lambda_- v_\mu^- v_\nu^- + \lambda \sum_{I=1}^{2}(-e_\mu^I e_\nu^I + e_\mu^{I+2} e_\nu^{I+2}) \,, \tag{B2}$$

where the three real parameters $\lambda_\pm$ and λ obey $\lambda_+ \lambda_- = \lambda^2$ and $(v_\mu^\pm, e_\mu^p)$, $p = 1, 2, 3, 4$, is an element of the coset $SO(5,1)/SO(1,1) \times SO(4)$ defined by

$$v^{\pm\mu} v_\mu^\pm = \eta^{\pm\pm} \,, \quad v^{\pm\mu} e_\mu^p = 0 \,, \quad e^{p\mu} e_\mu^q = \delta^{pq} \,, \tag{B3}$$

$$\eta^{+-} = \eta^{-+} = -1 \,, \quad \eta^{++} = \eta^{--} = 0 \,, \quad \epsilon^{-+pqrs} = \epsilon_{+-pqrs} = \epsilon^{pqrs} \,. \tag{B4}$$

If $\lambda_\pm \neq 0$ (the cases $\lambda_+ = \lambda = 0$ and $\lambda_- = \lambda = 0$ correspond to the infinitely boosted solutions discussed in the next appendix) we can use a boost to set $\lambda_+ = \lambda_- = \lambda$ such that

$$k_{\mu\nu} = -\lambda \eta_{\alpha\beta} v_\mu^\alpha v_\nu^\beta + \lambda \delta_{ab} u_\mu^a u_\nu^b \,, \tag{B5}$$

where (v_μ^α, u_μ^a), $\alpha = 0, 1, 2$, $a = 3, 4, 5$, is an element of the coset $SO(5,1)/SO(2,1) \times SO(3)$ obeying (16). Keeping the coset element fixed, any three–form can be expanded in this basis as follows

$$\omega_{\mu\nu\rho} = A u_{\mu\nu\rho}^3 + B_\alpha^a (u^2 v)_{a\,\mu\nu\rho}^\alpha + C_a^\alpha (uv^2)_{\alpha\,\mu\nu\rho}^a + D v_{\mu\nu\rho}^3 \,, \tag{B6}$$

where the coefficients together make up twenty real components and we have defined the following basis elements for three–forms:

$$u_{\mu\nu\rho}^3 = \epsilon_{abc} u_\mu^a u_\nu^b u_\rho^c \,, \quad (u^2 v)_{a\,\mu\nu\rho}^\alpha = \epsilon_{abc} u_{[\mu}^b u_\nu^c v_{\rho]}^\alpha \,, \tag{B7}$$

$$(uv^2)_{\alpha\,\mu\nu\rho}^a = \epsilon_{\alpha\beta\gamma} u_{[\mu}^a v_\nu^\beta v_{\rho]}^\gamma \,, \quad v_{\mu\nu\rho}^3 = \epsilon_{\alpha\beta\gamma} v_\mu^\alpha v_\nu^\beta v_\rho^\gamma \,. \tag{B8}$$

For our choice of Lorentz basis, the Hodge $\star$ acts as follows

$$\star u^3 = v^3 \,, \quad \star (uv^2)^{a,\alpha} = (u^2 v)^{a,\alpha} \,. \tag{B9}$$

Hence, we can write a general self–dual tensor in the form (B6) using the ten coefficients $A = D$ and $B^{a,\alpha} = C^{a,\alpha}$. In particular, we can apply this expansion to $h_{\mu\nu\rho}$ itself, which yields the following expression for $k_{\mu\nu}$:

$$k_{\mu\nu} = 2A^2(u_{\mu\nu}^2 - v_{\mu\nu}^2) + \frac{2}{9}\left[\left(B^2\delta_{ab} - 2B_{ab}^2\right)u_\mu^a u_\nu^b + \left(B^2\eta_{\alpha\beta} + 2B_{\alpha\beta}^2\right)v_\mu^\alpha v_\nu^\beta\right]$$

$$+\frac{4}{9}\epsilon_{abc}\epsilon_{\alpha\beta\gamma}B^{a,\alpha}B^{b,\beta}u_{(\mu}^c v_{\nu)}^\gamma \ , \tag{B10}$$

where we have written $B_{\alpha\beta}^2 = \delta^{ab}B_{a,\alpha}B_{b,\beta}$, $B_{ab}^2 = \eta^{\alpha\beta}B_{a,\alpha}B_{b,\beta}$ and $B^2 = \delta^{ab}B_{ab}^2 = \eta^{\alpha\beta}B_{\alpha\beta}^2$. However, by assumption the matrix $k_{\mu\nu}$ must be diagonal in this basis. This implies that there must exist real numbers X and Y such that

$$B_{ab}^2 = X\delta_{ab} \ , \quad B_{\alpha\beta}^2 = Y\eta_{\alpha\beta} \ . \tag{B11}$$

This implies that $B_{a,\alpha}$ has to vanish since it is an irreducible representation of $SO(3) \times SO(2,1)$. This can be verified explicitly by first taking the determinant of the equations in (B11) which shows that $X = Y < 0$, and then deducing from $B_{1,1}^2 = B_{2,2}^2 = Y < 0$ that $B_{a,1} = B_{a,2} = 0$ and hence $X = Y = 0$. It then follows from $B_{0,0}^2 = 0$ that also $B_{a,0} = 0$. Substituting $B_{a,\alpha} = C_{a,\alpha} = 0$ into Eq. (B6) gives

$$h_{\mu\nu\rho} = A\left(u_{\mu\nu\rho}^3 + v_{\mu\nu\rho}^3\right) \ . \tag{B12}$$

From this is follows that the matrix $m_{\mu\nu}$ in Eq. (B1) is given by

$$m_{\mu\nu} = (1 - 4\ell_p^6 A^2)u_{\mu\nu}^2 + (1 + 4\ell_p^6 A^2)v_{\mu\nu}^2 \ . \tag{B13}$$

After application of (B1) one finally obtains the expressions (13) and (14) for the non–linearly self–dual three–form and the open membrane metric provided one makes the following identification:

$$h = \frac{4A}{1 - 4\ell_p^6 A^2} \ . \tag{B14}$$

APPENDIX C: THE DECOUPLING LIMIT IN THE INFINITE MOMENTUM FRAME

In this appendix we discuss a modified version of the decoupling limit of Section IV where a boost parameter is scaled such that it becomes infinite in the limit. This results in a

decoupled string action of the form (45) where the constant background field strength is given by applying the infinite boost to the generic solution (13). The infinitely boosted field strength is most naturally described in terms of an $SO(5,1)/SO(1,1) \times SO(4)$ coset element (B3) and a self–dual two–form in the four–dimensional Euclidean space perpendicular to the boost direction.

<div align="center">

The $SO(5,1)/SO(1,1) \times SO(4)$ parameterisation of $\mathcal{H}_{\mu\nu\rho}$

</div>

Given a coset element (B3) an arbitrary three–form $h_{\mu\nu\rho}$ can be parameterised in terms of two anti–symmetric matrices F_{pq} and F'_{pq} and two vectors G_p and G'_p by giving its components with the respect to the frame as follows ($p = 1, 2, 3, 4$):

$$h_{+pq} = F^-_{pq} + F'^+_{pq} , \quad h_{-pq} = F^+_{pq} + F'^-_{pq} , \tag{C1}$$

$$h_{+-p} = G_p + G'_p , \quad h_{pqr} = \epsilon_{pqrs}(G_s - G'_s) , \tag{C2}$$

where $F^\pm_{pq} = \frac{1}{2}(F_{pq} \pm \epsilon_{pqrs}F_{rs})$ and $F'^\pm_{pq} = \frac{1}{2}(F'_{pq} \pm \epsilon_{pqrs}F'_{rs})$. A self–dual three–form is obtained by setting $F'_{pq} = G'_p = 0$. By an $SO(5,1)$ rotation we can fix a coset frame such that $G_p = 0$. In this adapted frame the self–dual three–form then has the expansion

$$h_{\mu\nu\rho} = 3v^+_{[\mu}e^p_\nu e^q_{\rho]}F^-_{pq} + 3v^-_{[\mu}e^p_\nu e^q_{\rho]}F^+_{pq} = 3v^+_{[\mu}F^-_{\nu\rho]} + 3v^-_{[\mu}F^+_{\nu\rho]} , \tag{C3}$$

where $F^\pm_{\mu\nu} = e^p_\mu e^q_\nu F^\pm_{pq}$. The local $SO(1,1) \times SO(4)$ symmetry can be fixed by taking $v^\pm_\mu = (1, \pm\hat{n})$, where $\hat{n}$ is a unit vector in $\mathbf{R}^5$, and choosing e^p_μ such that $F^\pm_{pq} = f_\pm(i\sigma^2 \oplus (\pm i\sigma^2))_{pq}$, where $f_\pm$ are two real parameters. This makes a total of ten independent degrees of freedom. From (B1) and (8) it follows that the non–linearly self–dual three–form and the open membrane metric are given in this parameterisation by

$$\mathcal{H}_{\mu\nu\rho} = \frac{12}{1 - 4\ell^{12}_p(F^+)^2(F^-)^2} \left[v^+_{[\mu}F^-_{\nu\rho]} + v^-_{[\mu}F^+_{\nu\rho]} - 2\ell^6_p(F^-)^2 v^+_{[\mu}F^+_{\nu\rho]} - 2\ell^6_p(F^+)^2 v^-_{[\mu}F^-_{\nu\rho]} \right] , \tag{C4}$$

$$G_{\mu\nu} = \frac{1}{\left(1 - 4\ell^{12}_p(F^+)^2(F^-)^2\right)^2} \left[\left(1 + 4\ell^{12}_p(F^+)^2(F^-)^2\right) g_{\mu\nu} \right.$$

$$\left. + 4\ell^6_p v^+_\mu v^+_\nu (F^-)^2 + 4\ell^6_p v^-_\mu v^-_\nu (F^+)^2 - 16\ell^6_p F^-_{\mu\rho}F^{+\rho}_\nu \right] . \tag{C5}$$

<div align="center">

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</div>

where we have defined $(F^{\pm})^2 = F^{\pm\mu\nu}F^{\pm}_{\mu\nu} = F^{\pm pq}F^{\pm}_{pq}$. Provided that $F^{\pm}_{\mu\nu} \neq 0$ this parameterisation is equivalent to the parameterisation in (13). An infinite boost along the direction $\hat{n}$ is obtained by setting $F^+_{\mu\nu} = 0$ (corresponding to the special case $\lambda_- = \lambda = 0$ mentioned under Eq. (B4)) and results in the expressions

$$\mathcal{H}_{\mu\nu\rho} = 12v^+_{[\mu}F^-_{\nu\rho]} , \tag{C6}$$

$$G_{\mu\nu} = g_{\mu\nu} + 4\ell^6_p v^+_\mu v^+_\nu (F^-)^2 . \tag{C7}$$

We notice that (C6) is actually linearly self–dual and the second term in (C7) drops out of the non–linear condition (6). More explicitly, in the infinite momentum frame the components of $\mathcal{H}$ are

$$\mathcal{H}_{+pq} = 4F^-_{pq} , \quad \mathcal{H}_{-pq} = 0 , \quad \mathcal{H}_{pqr} = \mathcal{H}_{+-p} = 0 . \tag{C8}$$

The Infinite Momentum Decoupling Limit

In order to define a decoupling limit leading to (45) with the field strength given by (C6), we first boost (13) along the 5 direction:

$$\begin{pmatrix} v^0 \\ u^5 \end{pmatrix} = \begin{pmatrix} \sqrt{1+a^2} & a \\ a & \sqrt{1+a^2} \end{pmatrix} \begin{pmatrix} \tilde{v}^0 \\ \tilde{u}^5 \end{pmatrix} , \tag{C9}$$

and then keep $\tilde{v}^0$ and $\tilde{u}^5$ fixed, while scaling according to (37) and

$$a \sim \epsilon^{-\gamma} a , \quad \gamma > 0 . \tag{C10}$$

The Wess–Zumino term now has weight $-\gamma - \Delta$, such that in the rescaled action (where the Wess–Zumino term is fixed) the kinetic energy of the parallel modes has weight $-2+\delta-1+\gamma+\Delta > 0$ and the kinetic energy of the perpendicular modes has weight $-2+\gamma+\Delta < 0$. The open membrane metric becomes

$$G_{\mu\nu} = \frac{(1+\sqrt{1+h^2\ell^6_p})^2}{4} \left[\delta_{ab} u^a_\mu u^b_\nu + \frac{1}{1+h^2\ell^6_p} \eta_{\alpha\beta} v^\alpha_\mu v^\beta_\nu \right. \tag{C11}$$

$$\left. + \frac{a^2 h^2 \ell^6_p}{1+h^2\ell^6_p} \left(v^0_\mu v^0_\nu + 2\sqrt{1+\frac{1}{a^2}} v^0_{(\mu} u^5_{\nu)} + u^5_\mu u^5_\nu \right) \right] , \tag{C12}$$

where we have dropped the tildes on v^0 and u^5. Requiring the leading components of this matrix to form a rank six matrix, we see that $a^2 h^2 \ell_p^6$ has to be finite, i.e. $\gamma + \Delta + \frac{3}{2}(\delta - 1) \leq 3$. Thus the conditions on $\gamma + \Delta$ are the same as the conditions on Δ given in (44). Since $h\ell_p^3 \to 0$, the result of the limit is the decoupled boundary Wess–Zumino term (45) with background field strength now given by the infinitely boosted solution (C6) provided we identify

$$v_\mu^+ = \frac{1}{\sqrt{2}}(v_\mu^0 + u_\mu^5) \;, \quad F_{\mu\nu}^- = \frac{ah}{\sqrt{2}}(-v_{[\mu}^1 v_{\nu]}^2 + u_{[\mu}^3 u_{\nu]}^4) \;. \tag{C13}$$

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